Problema 942. Suppose ABC is any triangle with angle bisectors AA', BB', and CC' meeting at the incenter I. Let $A'' = IA \cap B'C'$, $'' = IB \cap A'C'$, and $C'' = IC \cap A'B'$. Prove that

$$\frac{IA''}{A''A} + \frac{IB''}{B''B} + \frac{IC''}{C''C} = 1$$

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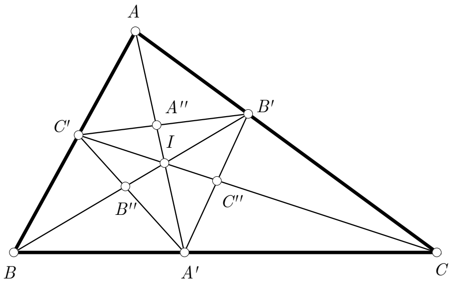

Let r, s, Δ be the inradius, the semiperimeter and the area of $\triangle ABC$ and let $a = BC$, $b = CA$, $c = AB$. By using the well known formulas

$$AB' = \frac{bc}{a+c} \qquad AC' = \frac{bc}{a+b}$$

$$AI = \frac{r}{\sin\frac{A}{2}} = \frac{2\Delta}{2s \cdot \sin\frac{A}{2}} = \frac{bc \cdot \sin A}{(a+b+c) \cdot \sin\frac{A}{2}} = \frac{2bc \cdot \cos\frac{A}{2}}{a+b+c}$$

we have

$$AA'' = \frac{2 \cdot AC' \cdot AB'}{AC' + AB'} \cdot \cos\frac{A}{2} = \frac{2bc \cdot \cos\frac{A}{2}}{2a+b+c} \qquad (1)$$

and

$$IA'' = AI - AA'' = \frac{2bc \cdot \cos\frac{A}{2}}{a+b+c} - \frac{2bc \cdot \cos\frac{A}{2}}{2a+b+c} = \frac{2abc \cdot \cos\frac{A}{2}}{(a+b+c)(2a+b+c)} \qquad (2)$$

From (1) and (2) it follows that

$$\frac{IA''}{A''A} = \frac{a}{a+b+c} \qquad (3)$$

By using (3) and its cyclic relations we get the desired result. □